Discipline: **Physics**

Subject: **Electromagnetic Theory**

Unit 22:

Lesson/ Module: **Motion in Nonuniform, Static Magnetic Fields**

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Contents

Learning Objectives:

From this module students may get to know about the following:

1. *Relativistic motion of charged particle in static but nonuniform magnetic field configurations.*
2. *Motion in a magnetic field with gradient perpendicular to the direction of the magnetic field leading to gradient drift.*
3. *Motion in magnetic field with curving lines of force leading to the curvature drift.*
4. *Motion when both gradient and curvature drift are present.*
5. *Motion along the direction of a slowly varying magnetic field and the idea of adiabatic invariance. Use of such a configuration as magnetic mirror.*

22. Motion in Nonuniform, Static Magnetic Fields - II

22.1 The Gradient Drift

We now study the nature of particle trajectories in a static but non-uniform magnetic field. Consideration of non-uniform magnetic fields is of interest in many areas, such as magneto-hydrodynamics, astrophysics and thermonuclear applications. Often, as in the case of interstellar and intergalactic space, the field is rather slowly varying, so that the problem can be tackled perturbatively. We know that for a uniform and static magnetic field, the trajectory of a particle is a helix with uniform motion along the direction of field and circular motion with certain radius of gyration in a plane perpendicular to the field. By slowly varying we mean the distance over which the field varies appreciably in magnitude or direction is large compared to the radius of gyration of the particle, so that over one gyration the field can be taken to be nearly constant locally. The motion is then one of gyration with a radius and frequency that depends on the "local" value of the field. In the next order of approximation, slow changes occur in the orbit of the particle which can be described as the "drift" of the guiding centre.

Let us first consider a spatial variation of the magnetic field $\vec{B}$ which is a gradient perpendicular to the direction of the field. Let the gradient at the point of interest be in the direction of the unit vector, $\hat{n}$, so that $\hat{n}.\vec{B}=0$. The particle has charge *q*, mass *m* and velocity $\vec{v}$. Then to first order, the gyration frequency can be written as [refer to the last module]

$$\vec{\omega}_B(\vec{x}) = \frac{q}{\gamma m}\vec{B}(\vec{x}) = \frac{q}{\gamma m}[\vec{B}_0 + (\frac{\partial B}{\partial \xi})_0(\vec{n}.\vec{x})] = \vec{\omega}_0[1+\frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0(\vec{n}.\vec{x})] \qquad (1)$$

Here γ is the usual Lorentz factor $1/\sqrt{1-\frac{v^2}{c^2}}$, $\vec{B}_0$ is the field at the guiding centre, B_0 is its magnitude, and $\vec{\omega}_0$ the gyration frequency at that point. Since the direction of the field is unchanged, the motion of the particle parallel to the field remains a uniform translation. Therefore we consider modifications in the motion perpendicular to the field only. Let $\vec{v}_0$ be the transverse velocity of the particle in the uniform field case and $\vec{v}_1$ the correction to it. Since the non-uniformity of the field is small, the magnitude of $\vec{v}_1$ is small compared to that of $\vec{v}_0$. Let us write

$$\vec{v}_\perp = \vec{v}_0 + \vec{v}_1 \qquad (2)$$

for the transverse component of the velocity of the particle. Then from the Lorentz force equation we obtain

$$\frac{d\vec{v}_\perp}{dt} = \frac{e}{\gamma m}\vec{v}_\perp \times \vec{B}(\vec{x}) = \vec{v}_\perp \times \omega_B(\vec{x})\,.$$

Or

$$\frac{d(\vec{v}_0 + \vec{v}_1)}{dt} = (\vec{v}_0 + \vec{v}_1)\times\vec{\omega}_0[1 + \frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0(\vec{n}.\vec{x})] \tag{3}$$

On comparing the zero and first order terms, we obtain

$$\frac{d\vec{v}_0}{dt} = \vec{v}_0 \times \vec{\omega}_0 \tag{4}$$

$$\frac{d\vec{v}_1}{dt} = [\vec{v}_1 + \vec{v}_0(\hat{n}.\vec{x}_0)\frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0]\times\vec{\omega}_0 \tag{5}$$

The first is the equation for the unperturbed motion. In the second only the first order terms are retained consistently, $\vec{v}_1 \times \vec{\omega}_0 \frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0(\vec{n}.\vec{x})$, being a second order term in perturbation, is ignored. .

The solution of the problem in a uniform and static magnetic field is well known and we have derived it earlier (Motion in Electric and Magnetic Fields – Part I). In the transverse direction the solution is

$$\vec{v} = a\omega_B\cos(\omega_B t)\hat{x} - a\omega_B\sin(\omega_B t)\hat{y} \tag{6}$$

$$\vec{x}(t) = \vec{X}_o + a\sin(\omega_B t)\hat{x} + a\cos(\omega_B t)\hat{y}\,. \tag{7}$$

On taking the cross product of the above two equations with $\vec{\omega}_B$, which is a vector along the *z*-direction, we have

$$\begin{aligned} \vec{v} &= -\vec{\omega}_B \times (\vec{x} - \vec{X}_0) \\ (\vec{x} - \vec{X}_0) &= \frac{1}{\omega_B{}^2}(\vec{\omega}_B \times \vec{v}) \end{aligned} \tag{8}$$

In the present case the unperturbed velocity is $\vec{v}_0$, the unperturbed position is $\vec{x}_0$ and the unperturbed frequency is $\vec{\omega}_0$. The centre of gyration of the unperturbed motion has been taken to be the origin, hence $\vec{X}_0 = 0$. On making the appropriate changes in the symbols we have

$$\vec{v}_0 = -\vec{\omega}_0 \times \vec{x}_0$$
$$\vec{x}_0 = \frac{1}{{\omega_0}^2}(\vec{\omega}_0 \times \vec{v}_0) \tag{9}$$

Next we use equation (9) to eliminate $(\vec{\omega}_0 \times \vec{v}_0)$ from equation (5) to get

$$\frac{d\vec{v}_1}{dt} = [\vec{v}_1 - \frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0(\hat{n}.\vec{x}_0)(\vec{\omega}_0 \times \vec{x}_0)] \times \vec{\omega}_0 \tag{10}$$

Had the additional factor $\frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0(\hat{n}.\vec{x}_0)(\vec{\omega}_0 \times \vec{x}_0) \times \vec{\omega}_0$ been absent, the solution would have been similar to the one for the zero order equation, viz., the oscillatory solution, $\vec{v}_1 = a\omega_B \cos(\omega_B t)\hat{x} - a\omega_B \sin(\omega_B t)\hat{y}$. Because of the presence of the inhomogeneous term, $\frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0(\hat{n}.\vec{x}_0)(\vec{\omega}_0 \times \vec{x}_0) \times \vec{\omega}_0$, the solution has an additional term with a nonzero average value:

$$\vec{v}_G = \langle \vec{v}_1 \rangle = \frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0 \vec{\omega}_0 \times < (\vec{x}_0)_\perp (\vec{n}.\vec{x}_0 > \tag{11}$$

The rectangular components of $(\vec{x}_0)_\perp$ oscillate sinusoidally with amplitude *a* and a phase difference of $\pi/2$ [look at equation (9)]. Hence only the component of $(\vec{x}_0)_\perp$ parallel to $\hat{n}$ contributes to the average. Thus

$$< (\vec{x}_0)_\perp (\vec{n}.\vec{x}_0 >= \frac{a^2}{2}\hat{n}\,. \tag{12}$$

Substituting this value into equation (11), we have for the *gradient drift velocity*

$$\vec{v}_G = \frac{a^2}{2}\frac{1}{B_0}(\frac{\partial B}{\partial \xi})_0(\vec{\omega}_0 \times \vec{n})\,. \tag{13}$$

Since $\frac{\partial B}{\partial \xi}$ is the normal gradient of magnetic field (in the direction of $\hat{n}$), $(\frac{\partial B}{\partial \xi})_0 \vec{n} = \vec{\nabla}_\perp B$. Also, $\vec{\omega}_0$ and $\vec{B}$ being parallel, $\vec{\omega}_0 B = \omega_0 \vec{B}$. So equation (13) can be written in an alternative dimensionless form, independent of the coordinates:

$$\frac{\vec{v}_G}{\omega_0 a} = \frac{a}{2B^2}(\vec{B} \times \vec{\nabla}_\perp B) \tag{14}$$

- From equation (14) it is evident that if the gradient of the field is small compared to the field, i.e., $a\left|\frac{\nabla B}{B}\right| << 1$, the drift velocity is small compared to the orbital velocity $(\omega_0 a)$. The particle spirals rapidly while its centre of rotation moves slowly in a direction normal to both $\vec{B}$ and gradient of *B*.

- The sense of drift is given by (14) for positively charged particles. That is, if the field is along the *z*- direction and its gradient along the *x*-direction, then the drift will be in the positive *y*-direction.

- For negatively charged particles ω_0 has a negative sign ($\vec{\omega}_0 = \frac{q\vec{B}}{\gamma mc}$) and hence the direction of the drift is opposite. **[See FIGURE, Fig12.3, Jackson Edition 2]**

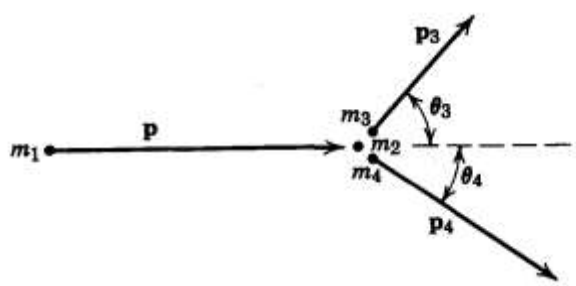

Fig. 12.3 Momentum vectors in laboratory for a two-body process.

- **Qualitative explanation:** The gradient drift can be understood qualitatively in the following way: As the particle circles around the guiding centre, it passes through regions of varying field. For half the cycle the field is greater than the average and for the other half it is less than the average field. Since the radius of gyration is inversely proportional to the field, for half the cycle the particle moves in a tighter than the average arc and for the other half in a broader than the average arc, producing a net drift in the transverse direction.

22.2 The Curvature Drift

The second type of field variation we consider is the curvature of lines of force. This type of variation also leads to a drift of the guiding centre, called *the curvature drift*. Consider a two dimensional field. To be specific, let the field be in the *x*-*y* plane. The unperturbed uniform field, $\vec{B}_0$, is along the *x*-axis. In such a field a particle spirals around the lines of force with radius of gyration, *a*, and velocity $\omega_B a$. For the field under consideration the lines of force are curved with a local radius of curvature *R* which is large compared to the radius of gyration. Therefore the nonuniformity can be regarded as a small perturbation to the unperturbed motion. **[Sea FIGURE, Fig 12.4(a) and (b) Jackson]**

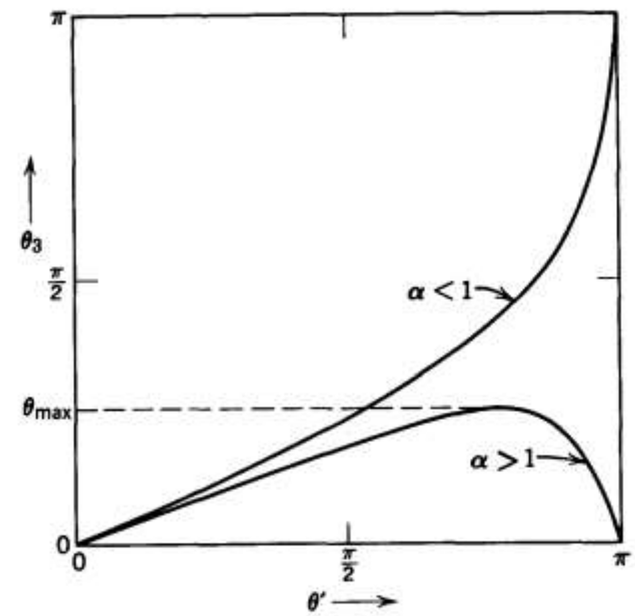

Fig. 12.4 Laboratory angle θ_3 of particle 3 versus center of momentum angle θ' for $\alpha < 1$ and $\alpha > 1$.

Physically the first order motion can be understood as follows: The particle tends to spiral around a field line, but the field line instead of being straight, curves off to a side. As far as the motion of the guiding centre is concerned, this is equivalent to a centrifugal acceleration of magnitude $v_{\|}^2/R$, which means a centrifugal force of magnitude $m\gamma v_{\|}^2/R$. The same effect would have been produced had there been an electric field, $\vec{E}_{eff} = \frac{\gamma m}{q}\frac{\vec{R}}{R^2}v_{\|}^2$. This effective field is in a direction transverse to the magnetic field. Since the radius of curvature is large compared to the radius of gyration, the effective electric field is small compared to the magnetic induction. The motion is therefore equivalent to that in crossed electric and magnetic fields with $E<cB$. This causes a drift with drift velocity $c\frac{\vec{E}\times\vec{B}}{B^2}$. Thus the curvature of the lines of force leads to a *curvature drift velocity*, $\vec{v}_c$, given by

$$\vec{v}_c = \frac{\vec{E}_{eff}\times\vec{B}}{B^2} = \frac{\gamma m}{q}\frac{\vec{R}\times\vec{B}_0}{R^2 B_0^{\,2}}v_{\|}^2 . \tag{15}$$

- We emphasize that no actual electric field is produced. The effect of a force $\vec{F}$ on the motion of a particle does not depend on the origin of that force. Thus the effect is the same had an actual electric force been present.

- The frequency of gyration is given by $\vec{\omega}_B = q\vec{B}_0/\gamma m$. Using this expression, $\vec{v}_c$ can be written in a more transparent form

$$\vec{v}_c = \frac{v_{\|}^2}{\omega_B R}\frac{\vec{R}\times\vec{B}_0}{RB_0} . \tag{16}$$

- The direction of the drift is specified by the vector product in which $\vec{R}$ is the radius vector *from* the effective centre of curvature to the position of the particle. Thus if the field is along the *x*-axis and the lines of field are curving upwards, so that the centre of curvature is along the positive *y*-direction, the drift velocity is along the *z* direction.

- This is true for a positively charged particle. For a negative charge the sign will be opposite because of the factor ω_B.

- The direction of drift is independent of the sign of $v_{||}$, since it depends on the square of the velocity.

22.2.1 An alternative derivation of curvature drift

A more straightforward derivation of the curvature drift can be given by the direct solution of the Lorentz force equation. If the centre of curvature is chosen as the origin of the system of coordinates, then in cylindrical coordinates, the magnetic field has only an azimuthal component, $B_\phi = B_0$.

Let us recall some of the relevant results from vectors in cylindrical coordinates. Write a vector $\vec{A}$ in both cylindrical and Cartesian coordinates:

$$\begin{aligned}\vec{A} &= A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \\ &= A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}\end{aligned} \tag{17}$$

Here $(\hat{\rho}, \hat{\phi}, \hat{z})$ are unit vectors along the three cylindrical coordinates. Then

$$\begin{aligned}A_\rho &= \cos(\phi)A_x + \sin(\phi)A_y \\ A_\phi &= -\sin(\phi)A_x + \cos(\phi)A_y \\ A_z &= A_z\end{aligned} \tag{18}$$

In particular

$$\vec{r} = \rho\hat{\rho} + z\hat{z} \tag{19}$$

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \tag{20}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z} \tag{21}$$

Here the dot (.) on a variable refers to time derivative. Since in a magnetic field the energy is constant, the Lorentz force equation $\dfrac{d\vec{p}}{dt} = \dfrac{q}{c}\vec{v} \times \vec{B}$ becomes

$$\gamma m\dot{\vec{v}} = q\vec{v}\times\vec{B} \tag{22}$$

or

$$\begin{aligned}(\ddot{\rho}-\rho\dot{\phi}^2)\hat{\rho}+(2\dot{\rho}\dot{\phi}+\rho\ddot{\phi})\hat{\phi}+\ddot{z}\hat{z} &= \frac{q}{\gamma m}(\dot{\rho}\hat{\rho}+\rho\dot{\phi}\hat{\phi}+\dot{z}\hat{z})\times B_0\hat{\phi} \\ &= (\dot{\rho}\hat{\rho}+\rho\dot{\phi}\hat{\phi}+\dot{z}\hat{z})\times\omega_B\hat{\phi}\end{aligned} \tag{23}$$

On comparing the coefficients of various unit vectors, $\hat{r},\hat{\phi},\hat{z}$, we obtain

$$\begin{aligned}&\ddot{\rho}-\rho\dot{\phi}^2 = -\omega_B\dot{z} \\ &2\dot{\rho}\dot{\phi}+\rho\ddot{\phi}^2 = 0 \\ &\ddot{z} = \omega_B\dot{\rho}\end{aligned} \tag{24}$$

We are looking for the lowest order solution for which $\rho=R$. The second equation then gives $\ddot{\phi}=0$, or $\dot{\phi}=$ constant=$v_{||}/R$. Substituting this in the first equation gives

$$\dot{z} = \frac{v_{||}^{\,2}}{\omega_B R} \tag{25}$$

- This is just the curvature drift velocity derived earlier. Note that this is an exact solution of the Lorentz force equation.

22.3 Combined curvature and gradient drifts

Consider the gradient drift that accompanies a curvature drift in cylindrical geometry. The magnetic induction is $\vec{B}=B_\phi\hat{\phi}=B_0\hat{\phi}$. In free space. $\vec{\nabla}\times\vec{B}=0$. From the z-component of this equation we obtain

$$(\vec{\nabla}\times\vec{B})_z = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho B_\phi)-\frac{\partial B_\rho}{\partial\phi}\right]=0 \Rightarrow \frac{\partial}{\partial\rho}(\rho B_\phi)=0 \Rightarrow B_\phi = \frac{B_0}{\rho} \tag{26}$$

where B_0 is some constant. Therefore

$$\vec{\nabla}_\perp B = \frac{\partial}{\partial\rho}(B_0/\rho)\hat{\rho} = -\frac{B_0}{\rho^2}\hat{\rho} = -\frac{B_\phi}{\rho^2}\vec{\rho} \tag{27}$$

Hence

$$\frac{\vec{B}\times\vec{\nabla}_{\perp}B}{B^2} = -\frac{\vec{B}\times(\frac{B\vec{\rho}}{\rho^2})}{B^2} = \frac{\vec{R}\times\vec{B}}{R^2 B} \tag{28}$$

Here we have used the lowest order solution, $\rho = R$, the radius of curvature. The gradient drift velocity $\vec{v}_G$ takes the form

$$\vec{v}_G = \frac{a^2\omega_B}{2}\frac{\vec{B}\times\vec{\nabla}_{\perp}B}{B^2} = \frac{1}{2}\frac{v_{\perp}{}^2}{\omega_B R}\frac{\vec{R}\times\vec{B}}{RB}\,. \tag{29}$$

where $v_{\perp} = a\omega_B$. We can now combine the two expressions, for $\vec{v}_G$ and $\vec{v}_C$ and obtain for the total drift velocity, $\vec{v}_D$

$$\vec{v}_D = \frac{v_{\parallel}{}^2 + \frac{1}{2}v_{\perp}{}^2}{\omega_B R}\frac{\vec{R}\times\vec{B}}{RB} \tag{30}$$

.For charged particles in thermal equilibrium, mean energy is $\frac{3}{2}kT$, and square of each component of velocity is *kT/m*. Here *k* is the Boltzmann constant and *T* the absolute temperature. Since $v_{\perp}$ has two components, $v_{\parallel}{}^2 \sim \frac{1}{2}v_{\perp}{}^2 \sim kT/m$.

- Hence the two terms contribute to the drifts with similar magnitudes.

- Substituting for the velocities in terms of the temperature from above and using the numerical value of the charge of electron and other constants, for a singly charged particle, we obtain

$$v_D(cm/\sec) = \frac{172\,T(^0K)}{R(m)B(gauss)}$$

- Particle drifts implied by the above are troublesome in thermonuclear machines designed to contain hot plasma. For the thermonuclear reaction to initiate, the hot plasma must attain "sufficiently high" temperature and stay confined for "sufficiently long" time. Because of the drifts the plasma will leak out of the walls and not remain confined.

22.4 Motion parallel to the Field - Adiabatic Invariance

The various motions we have discussed so far have been perpendicular to magnetic lines of force. These motions, caused by the electric fields or by the gradient or curvature of the magnetic field, arise because of the peculiarities of the Lorentz force, in particular, the fact that the force due to magnetic field is perpendicular to both the field and the velocity. These lead to "drifts" which at first sight seem to be counterintuitive. To complete our study of the motion in magnetic field, we now consider motion parallel to the magnetic field. It turns out that for slowly varying fields a powerful tool is the concept of *adiabatic invariants.*

An *adiabatic invariant* is a property of a physical system that stays constant when changes occur slowly. In thermodynamics, an adiabatic process is a change that occurs without heat flow, and slowly compared to the time to reach equilibrium. In an adiabatic process, the system is in equilibrium at all stages. Under these conditions, the entropy is constant.

In mechanics, an adiabatic change is a slow deformation of the Hamiltonian, where the fractional rate of change of the energy is much slower than the orbital frequency. The area enclosed by the different motions in phase space is the *adiabatic invariants*.

In quantum mechanics, an adiabatic change is one that occurs at a rate much slower than the difference in frequency between energy eigenstates. In this case, the system does not make transitions between energy states, so that the quantum number is an adiabatic invariant.

The old quantum theory was formulated by equating the quantum number of a system with its classical adiabatic invariant. This determined the form of the Bohr–Sommerfeld quantization rule: the quantum number is the area in phase space of the classical orbit.

In mechanics, adiabatic invariants are introduced by considering the action integrals of the system. Let a mechanical system be described by the generalized coordinates, q_i, and generalized momenta, p_i. If a particular generalized coordinate, q_i, is periodic then the *action integral* corresponding to it is defined as

$$J_i = \oint p_i dq_i$$

The integration is over one complete cycle of q_i. For a given mechanical system with given initial conditions, the action integrals are constants. If now the parameters of the system are changed in some way, and if the rate of change is slow compared to the rate of periodic motion, i.e., if the change is *adiabatic*, the action integrals are invariant. The change in the parameters will induce a change in the motion of the system in such a way that the value of the action integrals will remain the same. Such quantities are called *adiabatic invariants*.

22.4.1 Adiabatic invariants in charged particle motions

Let us consider the motion of a charged particle in a magnetic field. For a uniform and static magnetic field, $\vec{B}$, the transverse motion is periodic (circular in this case). The corresponding action integral is

$$J = \oint \vec{p}_\perp . d\vec{l} \qquad (31)$$

Here $d\vec{l}$ is the line element along the (circular) path of the particle and $\vec{p}_\perp$ is the transverse component of the *canonical momentum*

$$\vec{P} = \vec{p} + q\vec{A} \qquad (32)$$

Using this expression (32) into equation (31), we obtain

$$J = \oint \gamma m \vec{v}_\perp . d\vec{l} + q\oint \vec{A}.d\vec{l} \qquad (33)$$

Since $\vec{v}_\perp$ is parallel to $d\vec{l}$, $v_\perp = a\omega$ and $|d\vec{l}| = ad\theta$,

$$J = \oint \gamma m\omega a^2 d\theta + q\oint \vec{A}.d\vec{l}$$

In the first integral, integrating over θ gives a factor of 2π . In the second integral we use Stokes's theorem [$\oint \vec{A}.d\vec{l} = \int_S (\vec{\nabla}\times\vec{A}).\hat{n}ds$] to convert it into a surface integral. The result is

$$J = 2\pi\gamma m\omega a^2 + q\int_S \vec{B}.\hat{n}ds$$

Now the normal $\hat{n}$ to line element is anti-parallel to the direction of the magnetic field $\vec{B}$ is negative:

$$J = 2\pi\gamma m\omega a^2 - qB\pi a^2 = \pi\gamma m\omega a^2 = q(\pi a^2 B) \qquad (34)$$

where we have used the relation $\omega = \dfrac{qB}{\gamma m}$. The quantity $(\pi a^2 B)$ is the flux through the particle's orbit, and this quantity is an adiabatic invariant. In other words if the particle passes through a region in which the magnetic field is varying slowly in time or in space, the motion must be such that this flux remains constant. For example if the magnetic field increases the radius of gyration must decrease so that (Ba^2) remains unchanged. Using the relation between the transverse momentum, the orbit radijus and the magnetic moment of the current loop of the particle, this invariance can be phrased in many different ways:

$$Ba^2 \text{ or } p_\perp{}^2/B \text{ or } \gamma\mu$$

is an adiabatic invariant.

22.4.2 Gradient along the field

Let us now consider a simple situation in which a static magnetic field acts mainly in the *z*-direction, but has a small positive gradient in that direction as shown in the diagram below. **[See FIGURE, Fig 12.5, Jackson]** .

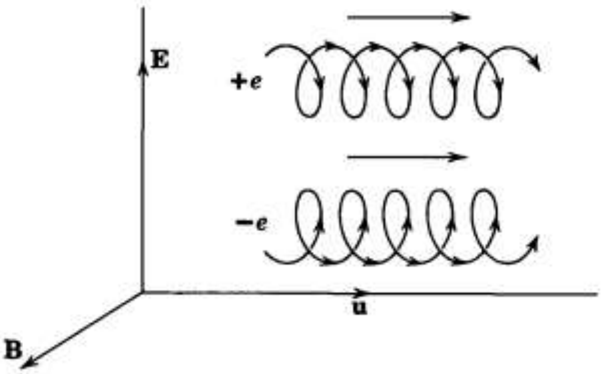

Fig. 12.5 **E × B** drift of charged particles in crossed fields.

Since the magnetic field is divergence free, i.e., $\vec{\nabla}.\vec{B}=0$, there is a small radial component of the field, in addition to the *z*-component. For simplicity we assume cylindrical symmetry. Suppose that a particle is spiraling around the *z*-axis in an orbit with a small radius. At $z = 0$, the field strength is B_0, the transverse velocity of the particle is $\vec{v}_{\perp,0}$ and the parallel component is $v_{||}$. Since the energy of the particle is fixed, at any position along the *z*-axis

$$v_{||}^{2}+v_{\perp}^{2}=v_{||,0}^{2}+v_{\perp,0}^{2}=v_{0}^{2} \qquad (35)$$

If we assume that the flux is an adiabatic invariant, then

$$\frac{v_{\perp}^{2}}{B}=\frac{v_{\perp,0}^{2}}{B_0} \qquad (36)$$

Combining the two equations above, we have

$$v_{||}^{2}=v_{0}^{2}-v_{\perp}^{2}=v_{0}^{2}-v_{\perp,0}^{2}\frac{B(z)}{B_0} \qquad (37)$$

- As the particle proceeds along the positive *z*-direction, the magnetic field increases, and hence, $v_{||}$ decreases. If *B(z)* increases enough, eventually the right hand side of the equation becomes zero. Also as *B(z)* increases, the radius of gyration decreases. This means that the particle spirals in an orbit with decreasing radius of gyration, converting more and more of its translational energy into rotational energy until its axial velocity vanishes. Then the particle turns around, still spiraling in the same sense, clockwise or anti-clockwise, and starts moving in the negative *z*-direction. The particle is *reflected* by the magnetic field as depicted in the diagram below. **[See FIGURE, Fig 12.6, Jackson]**

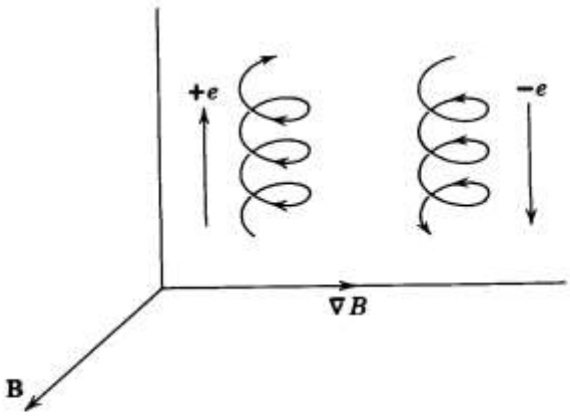

Fig. 12.6 Drift of charged particles due to transverse gradient of magnetic field.

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- We have thus what is called a *magnetic mirror*. The above equation is a consequence of the adiabatic invariance of $p_\perp{}^2/B$.
- In this particularly simple case the invariance follows directly from the Lorentz force equation. The field must be divergence free: $\vec\nabla.\vec B = 0$. In cylindrical coordinates, the divergence has the form

$$\vec\nabla.\vec B = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z} = 0 \qquad (38)$$

Since field has radial and axial components,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0 \qquad (39)$$

Provided $\frac{\partial B_z}{\partial z}$ does not vary much with ρ, we have

$$\rho B_\rho = -\int_0^\rho \rho\frac{\partial B_z}{\partial z}d\rho = -\frac{1}{2}\rho^2\frac{\partial B_z}{\partial z} \Rightarrow B_\rho = -\frac{1}{2}\rho\frac{\partial B_z}{\partial z} \qquad (40)$$

The Lorentz force equation is

$$\vec F = m\gamma\vec a = q\vec v\times\vec B \qquad (41)$$

The left hand side of this equation is

$$m\gamma\ddot{\vec a} = m\gamma[(\ddot\rho - \rho\dot\phi^2)\hat\rho + (2\dot\rho\dot\phi + \rho\ddot\phi)\hat\phi + \ddot z\hat z] \qquad (42)$$

The right hand side is

$$q\begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \dot{\rho} & \rho\dot{\phi} & \dot{z} \\ B_\rho & 0 & B_z \end{vmatrix} = q[\rho\dot{\phi}B_z\hat{\rho} + (\dot{z}B_\rho - \dot{\rho}B_z)\hat{\phi} - \rho\dot{\phi}B_\rho\hat{z}] \tag{43}$$

On equating the *z*-components, we have

$$m\gamma\ddot{z} = -q\rho\dot{\phi}B_\rho \Rightarrow \ddot{z} = -\frac{q}{m\gamma}\rho\dot{\phi}B_\rho \tag{44}$$

Substituting for B_ρ from equation (40)

$$\ddot{z} = \frac{q}{2m\gamma}\rho^2\dot{\phi}\frac{\partial B_z}{\partial z}. \tag{45}$$

In the present notation, $\rho^2\dot{\phi} = -(a^2\omega_B)_0 = -(v_{\perp 0}{}^2/\omega_{B0})$, so that

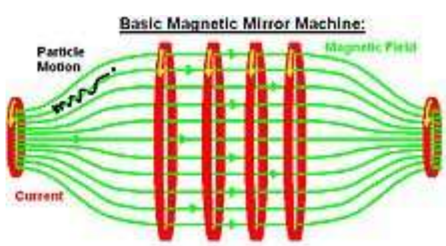
$$\ddot{z} = -\frac{v_{\perp 0}{}^2}{2B_0}\frac{\partial B(z)}{\partial z} \tag{42}$$

On multiplying both sides with $\dot{z}$ and integrating yields equation (37). Thus to first order in small quantities the constancy of flux follows directly from the equation of motion.

22.4.3 The magnetic mirror and magnetic bottle

magnetic mirror

Below is an image of an electron beam being reflected by the magnetic mirror effect.

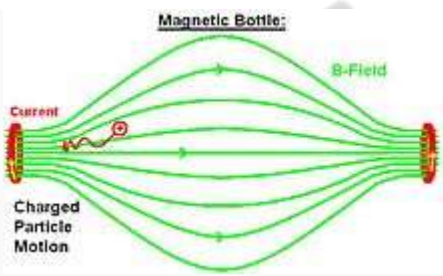


As we have seen above, a *magnetic mirror* is a configuration of magnetic field lines in which a charged particle is reflected from a region of high magnetic to low magnetic field. This mirror effect will only occur for particles within a limited range of velocity and angle of approach. Magnetic mirrors are made of specialized electromagnets designed to create a highly inhomogeneous field. Large magnetic mirrors have been used experimentally as a means of plasma confinement. One major application on which a lot of research is being carried out is to confine the hot, electrically charged plasma inside a fusion reactor to generate fusion power.

The concept of plasma confinement by using the idea of magnetic mirror was proposed in mid-1950s. By the late 1960s, magnetic mirror confinement was considered a viable technique for producing fusion. Magnetic mirrors also play an important role in other types of magnetic fusion energy devices such as tokamaks.

Magnetic mirrors also occur in nature. Electrons and ions in the magnetosphere, for example, will bounce back and forth between the stronger fields at the poles, leading to the Van Allen radiation belts.

Magnetic bottle



A *magnetic bottle* is two magnetic mirrors placed close together. For example, two parallel coils separated by a small distance, carrying the same current in the same direction will produce a magnetic bottle between them. The image shows how a charged particle will corkscrew along the magnetic fields inside a magnetic bottle. The particle can be reflected from the high field region and will be trapped. Unlike the full mirror machine which typically had many large rings of current surrounding the middle of the magnetic field, the bottle typically has just two rings of current. Particles near either end of the bottle experience a magnetic force towards the center of the region; particles with appropriate speeds spiral repeatedly from one end of the region to the other and back. Magnetic bottles can be used to temporarily trap charged particles. It is easier to trap electrons than ions, because electrons are so much lighter. This technique is used to confine very hot plasmas with temperatures of the order of 10^6 K.

In a similar way, the Earth's non-uniform magnetic field traps charged particles coming from the sun in doughnut shaped regions around the earth called the *Van Allen radiation belts*, which were discovered in 1958 using data obtained by instruments aboard the Explorer 1 satellite

Summary

1. *In this study of relativistic motion of charged particles in magnetic fields is continued. Motion in static but nonuniform magnetic fields is studied for various types of non-uniformities.*
2. *Motion in a magnetic field with gradient perpendicular to the direction of the magnetic field is described leading to gradient drift of the particle.*
3. *Next motion in magnetic field with curving lines of force is studied; and this leads to the curvature drift.*
4. *Both types of non-uniformities can be present together. It is found that the two terms contribute with similar magnitude.*
5. *Finally motion along the direction of a slowly varying magnetic field is studied. The concept of adiabatic invariance is introduced.*
6. *Motion along the magnetic field is analyzed by employing these adiabatic invariants. Use of such a configuration as magnetic mirror and magnetic bottle is discussed.*